Sound Radiation from Ducts for the Case of Low Mach Number

M.M. Wahbah* and W.C. Strahle†
Georgia Institute of Technology, Atlanta, Georgia.

A new method for rapidly solving duct end-plane impedance problems is presented. The method is based on deriving a new form of the Helmholtz integral formula expressing the normal derivative of the acoustic pressure at a boundary point. A set of integral expressions is obtained by satisfying the existing boundary conditions, and then solved using a general collocation method. The results obtained for the Levine-Schwinger problem agree well with the exact values. In the case of a duct flow having a temperature mismatch with the surroundings, the results show an increase in the magnitude of the reflection coefficient for the case of a cold core, and a decrease in its value for the case of hot core, as compared with the case of exhaust into a uniform medium. A phase change of between $\pi/2$ and π is nearly always achieved, which indicates the tendency towards maintaining a constant pressure at the exit plane. The method can easily be extended to arbitrary duct shapes, as long as they are axisymmetric; and the method consumes relatively little computer time.

I. Introduction

THE problem of core engine noise has been gaining strong recognition in recent years. As with other noise sources internal to a turbo-propulsion system, a major problem arises in determining the noise radiated to the surroundings, given the strength of the internal noise source. The problem is that of the impedance of the engine or duct termination. This problem arises in interpretation of the noise source strength when testing combustor rigs and in determining core noise strength from far-field sound measurements on engines.[1]

The reflection coefficient inside a duct, and hence the exit plane impedance, is obtained by solving the problem of sound radiation from a duct. This problem can be formulated as Wiener-Hapf problem.[2]

One technique used to solve this type of problem is called the Wiener-Hopf technique, which employs the analyticity of some functions of a complex variable in certain overlapping upper and lower half-planes, and the concept of analytic continuation. Levine and Schwinger[3] used this technique to solve the problem of sound radiation from a circular duct in the case of axially symmetric excitation. Carrier[4] extended this work to a duct immersed in a uniform flow, and the results can be expressed directly in terms of the Levine and Schwinger solution. Lansing et al.[5] extended this work further for the asymmetric modes.

The major difficulty in using this technique is that it requires the factorization of some functions of a complex variable into the quotient of two functions, one of which is analytic in the upper half-plane and the other is analytic in the lower half-plane; this is, in general, not easy to carry out. To solve this problem, Carrier[6] proposed the replacement of the function to be factored by another one which matches the exact one reasonably well and which may be easy to factor. Mani[7] adopted this procedure to solve the problem of refraction of acoustic duct waveguide modes. Savkar and Edelfelt[8] used the same technique to solve the problem of radiation of duct acoustic modes with flow mismatch, including the temperature mismatch as a special case, but they did not calculate the reflection coefficient for this latter case.

In addition to the factorization problem, the Wiener-Hopf technique is limited, at least from the practical point of view, to radiation from two parallel plates or circular cylinders.

The method presented in this paper employs a general collocation method, which is a remarkably simple technique for solving boundary-value problems.[9-12] It is based on approximating the exact solution by the sum of appropriate trail functions. The coefficients of the assumed expansion are obtained by satisfying the governing equations or some existing conditions at a certain number of points. The collocation method is a special case of the method of weighted residuals, where the weighting function is a delta function.

To check this technique the method is first applied to the Levine-Schwinger problem. Then it is extended to solve the case where the temperature inside the duct and its extension is different from that of the outside medium. The second model approximates an actual jet engine at low exit Mach number conditions and the experimental set up in Ref. 1. The computations were carried out with the main objective to determine the reflection coefficient required for the combustion noise research in Ref. 1. It should be noted that the method can be used to determine the complete sound field.

II. Levine-Schwinger Problem

New Formulation of the Problem

In this section we investigate the sound radiation from a circular semi-infinite duct immersed in a quiescent uniform medium. The duct walls are perfectly rigid and of negligible thickness. We consider a plane wave moving toward the open end. If the frequency of the incident wave is less than some value, only plane waves can propagate in the pipe, and part of the incident energy is returned back in a reflected wave of the same type, as shown in Fig. 1.

Under the usual assumptions of linear acoustics, the equations governing the propagation of simple harmonic disturbances are

$$\nabla^2 p + k^2 p = 0 \tag{1}$$

$$v = -(i/\rho\omega)\nabla p \tag{2}$$

where $k=\omega/c$, and $p \exp(-i\omega t)$ and $v \exp(-i\omega t)$ are the acoustic pressure and velocity, respectively. Although this model represents the case of zero mean velocity, it can be considered as an approximation for the case of low Mach number. For disturbances having a general time dependence,

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*Post Doctoral Fellow. Member AIAA.

†Regents' Professor. Associate Fellow AIAA.

the analysis remains the same if we replace p and v by the Fourier transform of the corresponding variable, in which case ω is the transform parameter.

Equation (1) is satisfied at any point in the domain V surrounded by the surface S (Fig. 1). The surface S is made up of the inner and the outer surfaces of the duct, S_1 and S_2, the disk S_3, and the spherical surface S_4. The solution should satisfy the condition of zero normal velocity at any point on the duct surface, namely

$$\partial p(s)/\partial n_s \tag{3}$$

where n_s is the unit outward normal at s. The value of p on S_3, as L goes to infinity, is equal to the sum of the incident and the reflected waves. Besides, the pressure must satisfy Sommerfeld's radiation condition on S_4 as a goes to infinity, this condition providing the uniqueness of the solution.[13] In view of the complexity of the boundary, an integral formulation is suggested. The tool for this purpose is the free space Green's function

$$G(s,q) = \exp(ikD(s,q))/4\pi D(s,q)$$

with $D(s,q)$ denoting the distance between any two points s and q in V. Using Green's theorem, the pressure at any point q is given by

$$p(q) = \int_S [G(s,q)n_s \cdot \nabla_s p(s) - p(s)n_s \cdot \nabla_s G(s,q)]\mathrm{d}S_s \tag{4}$$

The integral in Eq. (4) can be written as the sum of integrals over S_1, S_2, S_3, and S_4. The contributions from S_3 and S_4 vanish as L and a go to infinity. Equation (4) then becomes

$$p(q) = \int_{S_1+S_2}\left[G(s,q)\frac{\partial}{\partial n_s}p(s) - p(s)\frac{\partial}{\partial n_s}G(s,q)\right]\mathrm{d}S_s \tag{5}$$

Applying the boundary condition of zero normal velocity on the duct surface, Eq. (3), we get

$$p(q) = -\int_{S_1} p_1(s)\frac{\partial}{\partial n_s}G(s,q)\mathrm{d}S_s - \int_{S_2} p_2(s)\frac{\partial}{\partial n_s}G(s,q)\mathrm{d}S_s \tag{6}$$

where $p_1(s)$ is the acoustic pressure at the point s on S_1 and $p_2(s)$ is its value at the corresponding point on S_2.

It should be noted that the unit normal n_s at some point on the outer surface of the duct, S_2, is opposite in direction to n_s at the corresponding point on the inner surface S_1 (Fig. 1). To facilitate the manipulation of the above equation, we use the cylindrical coordinates (r_s, Φ_s, z_s) to specify any point s and denote the unit vectors in the directions of the coordinate axes at this point by $(e_{rs}, e_{\phi s}, e_{zs})$. Then

$$n_s = -e_{rs} \text{ if } s \text{ lies on } S_2$$

$$n_s = e_{rs} \text{ if } s \text{ lies on } S_1$$

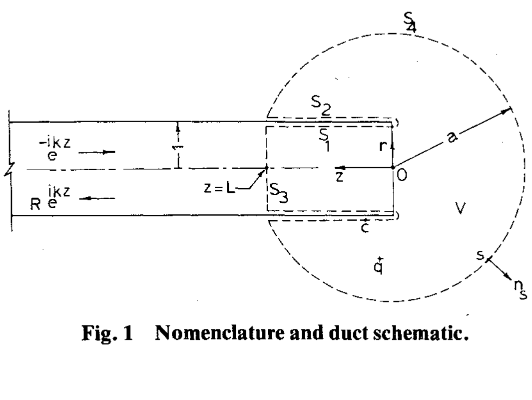

Fig. 1 Nomenclature and duct schematic.

Also,

$$D^2(s,q) = r_s^2 + r_q^2 - 2r_s r_q \cos(\phi_s - \phi_q) + (z_s - z_q)^2$$

Noting that $p(s)$ is independent of ϕ_s, and denoting the pressure discontinuity $p_1(z_s) - p_2(z_s)$ across the duct surface by $P(z_s)$, Eq. (6) becomes

$$p(q) = -\int_0^\infty P(z_s)K(s,q)dz_s \tag{7}$$

where

$$K(s,q) = \int_0^{2\pi}\left[\frac{\partial}{\partial r_s}G(s,q)\right]_{r_s=1}\mathrm{d}\Phi_s \tag{8}$$

Equation (7) confirms the assertion that the entire acoustic field is completely determined by its boundary values on the duct surface. We proceed now to determine $P(z_s)$.

integrals. The limits of these integrals are not equal to the integrals at the limit owing to the singularity in the normal derivative of the Green's function. This limiting process leads to the following surface Helmholtz integral formula

$$\frac{1}{2}p(c) = \int\left[G(s,c)\frac{\partial}{\partial n_s}p(s) - p(s)\frac{\partial}{\partial n_s}G(s,c)\right]\mathrm{d}S_s \tag{9}$$

Equation (9) can be derived from a consideration of properties of single- and double-layer distributions of potential theory;[14] or directly from Green's theorm.[15] Cauchy principal values should be used in interpreting the integrals occurring in this equation.

Similary, we obtain by differentiation of Eq. (4) and, taking the limit as q goes to c

$$\frac{1}{2}\frac{\partial}{\partial n_c}p(c) = \int\frac{\partial G(s,c)}{\partial n_c}\frac{\partial p(s)}{\partial n_s}\mathrm{d}S_s - \lim_{q\to c}\frac{\partial}{\partial n_q}\int p(s)\frac{\partial}{\partial n_s}G(s,q)\mathrm{d}S_s \tag{10}$$

where $\partial/\partial n_q$ denotes the derivative along a line passing through c in the direction of the outward normal. The integral resulting from differentiation under the integral sign, together with the direct substitution $q=c$, will diverge. Stallybrass[16] gave an interpretation of the limiting operation involved in the second integral, in Eq. (10), where $G(s,c)$ may be any fundamental singularity of the Helmholtz equation, $H(s,c)$, and not necessarily the free space Green's function $G(s,c)$. This interpretation is the key factor in the development of the present new formulation. The result involves only convergent integrals, at least in the Cauchy principal value sense, and we mention it here for completeness

$$\begin{aligned}\lim_{q\to c}\frac{\partial}{\partial n_q}\int_S p(s)\frac{\partial H(s,q)}{\partial n_s}\mathrm{d}S_s &= \int_S p(s)(n_c\cdot n_s)\nabla_c\cdot\nabla_s H(s,c)\mathrm{d}S_s \\ &+ \int_S p(s)(n_c\times n_s)\cdot[\nabla_c\times\nabla_s H(s,c)]\mathrm{d}S_s \\ &+ \int_S [n_c\times\nabla_c H(s,c)]\cdot[n_s\times\nabla_s p(s)]\mathrm{d}S_s\end{aligned} \tag{11}$$

For the case of $H(s,c) = G(s,c)$ the contribution from the second integral vanishes and with

$$\nabla_c\cdot\nabla_s G(s,c) = k^2 G(s,c)$$

Eq. (10) becomes

$$\frac{1}{2}\frac{\partial}{\partial n_c}p(c)=\int\frac{\partial G(s,c)}{\partial n_c}\frac{\partial p(s)}{\partial n_s}\,dS_s-\int\{k^2(n_c\cdot n_s)p(s)G(s,c)$$
$$+[n_c\times\nabla_c G(s,c)]\cdot[n_s\times\nabla_s p(s)]\}dS_s \quad (12)$$

Equation (12) is a new form of the Helmholtz integral formula which has not been used before in solving acoustics problems.

The integrals in Eqs. (9) and (12) are the sum of their values on S_1, S_2, S_3, and S_4. If the point c is confined to the finite part of the duct surface, then the contributions from S_3 and S_4 to $p(c)$ and $\partial p(c)/\partial n_c$ vanish as a and L go to infinity. [17]

Finally, we obtain the following expressions for the acoustic pressure and its normal derivative at any point c on the duct surface

$$\frac{1}{2}p(c)=\int_{S_1+S_2}[G(s,c)\frac{\partial p(s)}{\partial n_s}-p(s)\frac{\partial}{\partial n_s}G(s,c)]dS_s \quad (13)$$

$$\frac{1}{2}\frac{\partial p(c)}{\partial n_c}=\int_{S_1+S_2}\Big\{\frac{\partial G(s,c)}{\partial n_c}\frac{\partial p(s)}{\partial n_s}-k^2(n_c\cdot n_s)p(s)G(s,c)$$
$$-[n_c\times\nabla_c G(s,c)]\cdot[n_s\times\nabla_s p(s)]\Big\}dS_s \quad (14)$$

where the integration is performed on S_1 and S_2 only. In the present section we will use only Eq. (14), but both Eqs. (13) and (14) are necessary for use in Sec. III.

It should be noted that the direct substitution of $\partial p/\partial n=0$, on the duct surface, in Eq. (13) is not sufficient to take care of this boundary condition, and the resulting integral equation does not necessarily imply this condition. We should use Eq. (14) for this purpose.

Applying the boundary condition, Eq. (3), of zero normal velocity on the duct surface in Eq. (14), we get

$$0=\int_{S_1+S_2}\{k^2(n_c\cdot n_s)p(s)G(s,c)$$
$$+[n_c\times\nabla_c G(s,c)]\cdot[n_s\times\nabla_s p(s)]\}dS_s \quad (15)$$

Splitting this integral and using the relation

$$e_{rs}\times\nabla_s p(s)=-\frac{\partial P(s)}{\partial z_s}e_{\phi s}$$

Eq. (15) becomes

$$\int_0^{2\pi}d\Phi_s\int_0^{\infty}\{k^2P(s)\cos(\Phi_s-\Phi_c)G(s,c)$$
$$+\frac{\partial P(s)}{\partial z_s}\cos(\Phi_s-\Phi_c)\frac{\partial G(s,c)}{\partial z_c}\}dz_s=0 \quad (16)$$

Choosing the point c as a reference for the angular integration, and noting that $P(s)$ is independent of Φ_s, we obtain the following integro-differential equation for the pressure discontinuity across the duct surface

$$\int_0^{\infty}\Big\{k^2P(z_s)G_1(z_s,z_c)+\frac{dP(z_s)}{dz_s}G_2(z_s,z_c)\Big\}dz_s=0;\quad z_c>0 \quad (17)$$

where

$$G_1(z_s,Z_c)=\int_0^{\pi}\cos\Phi\frac{e^{ikD}}{D}\,d\Phi \quad (18)$$

$$G_2(z_s,z_c)=(z_c-z_s)\int_0^{\pi}\cos\Phi\frac{e^{ikD}}{D^2}(ik-1/D)d\Phi \quad (19)$$

and D is the distance between the points s and c which is given by

$$D=[(z_s-z_c)^2+2(1-\cos\Phi)]^{1/2} \quad (20)$$

Equation (17) can be solved using a general collocation method. We approximate $P(z)$ by a sum of trial functions and determine the coefficients involved in the expansion by satisfying the integral equation at some number of collocation points z_c. The selection of the trial set of functions and the choice of the points z_c are the most important steps in any procedure of this type. [9]

We observe that the radiation field is derived principally from the surface discontinuity of the pressure on a section of the duct, terminating at the open end, and whose linear dimension is comparable to the wavelength. At low frequencies, the length of this section is large compared to the transverse dimension of the duct, and the radiation field can be accurately derived from the asymptotic form of the pressure within the duct. Hence

$$[P(z)]_{k\to o}=e^{-ikz}+R\,e^{ikz} \quad (21)$$

where R is the reflection coefficient inside the duct. Guided by this fact, we assume the trial expression for $P(z)$, throughout the frequency range of purely plane-wave mode propagation ($k<3.832$) [18] as

$$P(z)=e^{-ikz}+R\,e^{ikz}+\sum_n a_n f_n(kz) \quad (22)$$

where f_n are the elements of the trail family and a_n are the unknown coefficients of expansion. In general, f_n and a_n are complex quantities.

The choice of Laguerre functions, as trial functions, is suggested for the following reasons:

(1) The Laguerre functions constitute a complete set of orthogonal functions over the interval from $z=0$ to $z=\infty$; hence, they are capable of expressing any reasonably well-behaved function defined over this range.

(2) These functions are real functions of their own variable and are defined by the recurrence relationship [19]

$$L_{n+1}(z)=(2n+1-z)L_n(z)-n^2L_{n-1}(z)$$
$$L_0(z)\;=e^{-z/2}$$
$$L_1(z)\;=e^{-z/2}(1-z) \quad (23)$$

Consequently, we notice that these functions decay exponentially as z increases, resulting in the reduction of the computation time for the evaluation of the infinite integrals in Eq. (17).

Practically, the series is truncated to contain only a finite number of terms, and Eq. (22) becomes

$$P(z)=e^{-ikz}+\sum_{n=1}^{N}A_nF_n(z) \quad (24)$$

where

$$A_1=R;\qquad F_1(z)=e^{ikz}$$
$$A_n=a_n;\qquad F_n(z)=L_{n-2}(kz)\quad(n\geq 2)$$

Substitution of Eq. (24) into Eq. (17) yields

$$\sum_{n=1}^{N}A_n\Big\{\int_0^{\infty}\Big[k^2F_n(z_s)G_1(z_s,z_c)+\frac{dF_n(z_s)}{dz_s}G_2(z_s,z_c)\Big]dz_s\Big\}$$
$$=-\int_0^{\infty}[k^2e^{-ikz_s}G_1(z_s,z_c)-ike^{-ikz_s}G_2(z_s,z_c)]dz_s\quad(z_c>0) \quad (25)$$

By satisfying the above equation at $(N-1)$ collocation points, to be chosen according to guidelines discussed later, and requiring the expansion to satisfy the condition of zero pressure discontinuity at the open end $z=0$, namely

$$\sum_{n=1}^{N} A_n F_n(0) = -1 \tag{26}$$

we obtain a set of N simultaneous linear algebraic equations. This set may be conviently written in the matrix notation

$$XA = Y \tag{27}$$

where A is the unknown coefficient column-matrix.

The integrals in Eqs. (25) are singular double integrals since G_1 and G_2 are themselves integrations with respect to φ_s.

Away from the singular point, the integrals in Eq. (25) can be evaluated numerically by using any numerical integration quadrature formula to evaluate the outer z_s integration. In doing this, the values of the G's, at each point z_s used in the outer z_s integration, are obtained numerically by evaluating the integrals in Eqs. (18) and (19) using the same or another quadrature formula. As the point z_s approaches the point z_c, the integrands in Eq. (25), being functions of z_s, give rise to different types of singularities. Smaller step sizes should be used in this region. The contribution from the final two steps, adjacent to z_c, is computed by first determing the type of singularity at this point and then using an equivalent one-step integration formula in a manner consistent with the definition of the Cauchy principle value.[17]

Finally the solution of the set of Eqs. (27) leads to the knowledge of the reflection coefficient R along with the rest of the expansion coefficients.

Results and Discussion

At low frequencies ($k \ll 1$), the pressure discontinuity across the duct surface is given by Eq. (21). By requiring this expression to satisfy the condition of pressure continuity at the open end, we get

$$R = -1$$

which is the exact value of R at zero frequency. On the other hand, it is difficult to analytically obtain the limiting value of R as k goes to zero, by considering the full system of Eqs. (27).

Several trials were made for a fixed wave number k and a fixed number of trail functions using different sets of collocation points. The convergence to the exact value of the reflection coefficient, as the number of trial functions increased, was first achieved with the set of collocation points chosen on the section of the duct whose length is equal to the wavelength of the incident wave and terminating at the open end. The convergence behavior of the trial series expansion was less satisfactory when the collocation points were distributed on a smaller section, terminating at the open end, and the answers were significantly in error when this section was smaller than half the wavelength of the incident wave. Although the convergence behavior of the trial series expansion was getting better as the length of this section was increased, the number of terms in the expansion required for convergence to the exact value of R was also increasing. In summary, the optimum results were obtained with the collocation points lying within the first wavelength of the duct.[17] This conclusion is in agreement with the fact that the radiation field is derived principally from the surface discontinuity of the pressure on that section.

The larger the frequency value, the larger was the number of terms in the trial expansion required for obtaining the reflection coefficient, with some fixed percentage of error. In fact, it was sufficient to use only the first three Laguerre functions besides the incident and reflected waves, to obtain the value of $|R|$ with less than 5 percent error for $k=1.0$.

The results are shown in Table 1 along with the corresponding exact values. The computed values were obtained with $N=8$ and with the collocation points equally distributed in the interval $0 < z_c \leq \lambda$.

As a concluding point of interest, a few comments on the infinite velocity at the duct periphery ($r=1$, $z=0$) are deemed appropriate. Studies on diffraction of acoustic waves from a knife-edge and on potential flows around sharp projecting edges show that the velocity at the edge is infinite. This is a necessary consequence of the assumed irrotational character of the motion.[20] Levine and Schwinger[3] showed that the axial and the radial velocities have square-root singularities at the duct periphery as z goes to 0^+ or 0^-, respectively. It follows that $dP(z_s)/dz_s$ in Eq. (17), being related to the component of particle velocity in the axial direction, on the duct surface, has a square-root singularity at $z_s=0$. Despite the existence of this singularity, the answers resulting from satisfying the integral Eqs. (17), using a series expansion which is continuous at $z=0$, are in good agreement with the exact values. This success is due to the fact that this singularity is an integrable one.

Finally, the computer time used to obtain the whole set of expansion coefficients for every value of k was about 6 seconds on CDC Cyber 74.

III. Extension of the Method to a Duct with Density Mismatch

Formulation of the Problem

In this problem the medium inside the duct and its cylindrical extension has a temperature which is different from the outside medium, and the mean velocity is zero everywhere. The present model can still be considered as an approximation for the case of low Mach number. The definition sketch for this problem is shown in Fig. 2.

The acoustic pressure is governed by

$$\nabla^2 p_1 + k_1^2 p_1 = 0 \text{ in region (1)} \tag{28}$$

$$\nabla^2 p_2 + k_2^2 p_2 = 0 \text{ in region (2)} \tag{29}$$

where

$$k_1 = \omega / c_1$$

$$k_2 = \omega / c_2$$

Here, and from now on, subscript 1 denotes quantities in region (1) and the subscript 2 denotes quantities in region (2).

The steady-state static pressures in the two regions are equal, and consequently, the ratio of the wave numbers, for perfect gases and equal specific heat ratios, becomes

$$k_2/k_1 = \sqrt{T_1/T_2} = \sqrt{\rho_2/\rho_1} \tag{30}$$

where ρ and T are the steady-state density and temperature, respectively.

The developments of Sec. II can be repeated, almost identically, separately in each of the two regions to obtain the acoustic pressure and its normal derivative at any point on S_1

Table 1 Reflection coefficients for several k

$R=\lvert R \rvert e^{i\alpha}$				
k	$\lvert R \rvert$ Exact	$\lvert R \rvert$ Computed	α Exact	α Computed
0.5	0.90	0.90	−2.56	−2.77
1.0	0.70	0.71	−2.10	−2.30
2.0	0.35	0.36	−1.46	−1.67
3.0	0.16	0.16	−1.28	−1.48

that in the present problem, both S_1 and S_2 extend from $z=-\infty$ to $z=\infty$. The results, for an arbitrary point c on S_i $(i=1,2)$ are

$$\frac{1}{2}p_i(c)=\int_{S_i}\left[G_i(s,c)\frac{\partial p_i(s)}{\partial n_s}-p_i(s)\frac{\partial}{\partial n_s}G_i(s,c)\right]\mathrm{d}S_s \tag{31}$$

$$\frac{1}{2}\frac{\partial p_i(c)}{\partial n_c}=\int_{S_i}\left\{\frac{\partial G_i(s,c)}{\partial n_c}\frac{\partial p_i(s)}{\partial n_s}-k_i^2(n_c\cdot n_s)p_i(s)G_i(s,c)-[n_c\times\nabla_c G_i(s,c)]\cdot[n_s\times\nabla_s p_i(s)]\right\}\mathrm{d}S_s \tag{32}$$

where $p_i(s)$ is the acoustic pressure at any arbitraty point s on S_i and $G_i(s,c)$ is the Green's function $G(s,c)$ with $k=k_i$.

The solution should satisfy the following boundary conditions:

a) Zero normal velocity on the inside surface of the duct

$$\frac{\partial p_1(z_c)}{\partial r_c}=0 \quad (z_c>0,\ r=1) \tag{33}$$

b) Zero normal velocity on the outside surface of the duct

$$\frac{\partial p_2(z_c)}{\partial r_c}=0 \quad (z_c>0,\ r=1) \tag{34}$$

c) Continuity of acoustic pressure on the duct extension

$$p_1(z_c)=p_2(z_c) \quad (z_c\leq 0,\ r=1) \tag{35}$$

d) Continuity of the radial acoustic particle velocity (which is equivalent to the continuity of the radial displacement in this problem) on the duct extension

$$\frac{1}{\rho_1}\frac{\partial p_1(z_c)}{\partial r_c}=\frac{1}{\rho_2}\frac{\partial p_2(z_c)}{\partial r_c} \quad (z_c<0,\ r=1)$$

Using Eq. (30) this condition can be written as

$$\frac{\partial p_1(z_c)}{\partial r_c}=\left(\frac{k_2}{k_1}\right)^2\frac{\partial p_2(z_c)}{\partial r_c} \quad (z_c<0,\ r=1) \tag{36}$$

Substituting Eqs. (31) and (32) into Eqs. (33)-(36), and after some manipulations, we obtain the following four integral formulas expressing the above conditions

$$\int_{-\infty}^{0}\frac{\partial p_-(z_s)}{\partial r_s}G_{4,1}(z_s,z_c)\mathrm{d}z_s-\int_{-\infty}^{0}\left[k_1^2p_-(z_s)G_{1,1}(z_s,z_c)+\frac{\partial p_-(z_s)}{\partial z_s}G_{2,1}(z_s,z_c)\right]\mathrm{d}z_s-\int_0\left[k_1^2p_i(z_s)G_{1,1}(z_s,z_c)+\frac{\partial p_i(z_s)}{\partial z_s}G_{2,1}(z_s,z_c)\right]\mathrm{d}z_s=0 \quad (z_c>0) \tag{37}$$

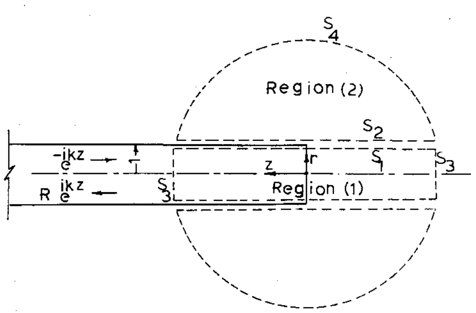

Fig. 2 Duct with density mismatch.

$$\left(\frac{k_2}{k_1}\right)^2\int_{-\infty}^{0}\frac{\partial p_-(z_s)}{\partial r_s}G_{4,2}(z_s,z_c)\mathrm{d}z_s-\int_{-\infty}^{0}\left[k_2^2p_-(z_s)G_{1,2}(z_s,z_c)+\frac{\partial p_-(z_s)}{\partial z_s}G_{2,2}(z_s,z_c)\right]\mathrm{d}z_s-\int_{-\infty}^{0}\left[k_2^2p_0(z_s)G_{1,2}(z_s,z_c)+\frac{\partial p_0(z_s)}{\partial z_s}G_{2,2}(z_s,z_c)\right]\mathrm{d}z_s=0$$
$$(z_c>0) \tag{38}$$

$$\int_{-\infty}^{0}\frac{\partial p_-(z_s)}{\partial r_s}\left[G_{3,1}(z_s,z_c)+\left(\frac{k_2}{k_1}\right)^2G_{3,2}(z_s,z_c)\right]\mathrm{d}z_s-\int_{-\infty}^{0}p_-(z_s)\left[G_{4,1}(z_s,z_c)+G_{4,2}(z_s,z_c)\right]\mathrm{d}z_s-\int_0^{\infty}p_i(z_s)G_{4,1}(z_s,z_c)\mathrm{d}z_s-\int_0^{\infty}p_o(z_s)G_{4,2}(z_s,z_c)\mathrm{d}z_s=0$$
$$(z_c<0) \tag{39}$$

$$\int_{-\infty}^{0}\frac{\partial p_-(z_s)}{\partial r_s}\left[G_{4,1}(z_s,z_c)+G_{4,2}(z_s,z_c)\right]\mathrm{d}z_s-\int_{-\infty}^{0}\left\{k_1^2p_-(z_s)\left[G_{1,1}(z_s,z_c)+G_{1,2}(z_s,z_c)\right]+\frac{\partial p_-(z_s)}{\partial z_s}\left[G_{2,1}(z_s,z_c)+\left(\frac{k_1}{k_2}\right)^2G_{2,2}(z_s,z_c)\right]\right\}\mathrm{d}z_s-\int_0^{\infty}\left[k_1^2p_i(z_s)G_{1,1}(z_s,z_c)+\frac{\partial p_i(z_s)}{\partial z_s}G_{2,1}(z_s,z_c)\right]\mathrm{d}z_s-\left(\frac{k_1}{k_2}\right)^2\int_0^{\infty}\left[k_2^2p_0(z_s)G_{1,2}(z_s,z_c)+\frac{\partial p_0(z_s)}{\partial z_s}G_{2,2}(z_s,z_c)\right]\mathrm{d}z_s=0 \quad (z_c<0) \tag{40}$$

where p_i, p_0, and p_- are the acoustic pressures on the inside surface of the duct, outside surface of the duct, and on the duct extension part of S_1, respectively, and

$$\frac{\partial p_-(z_s)}{\partial r_s}=\left[\frac{\partial p_1(s)}{\partial r_s}\right]_{r=1,\,z=z_s} \tag{41}$$

$G_{1,i}$ and $G_{2,i}$ $(i=1,2)$ are obtained from Eqs. (18), (19) by substituting $k=k_i$. Furthermore

$$G_{3,i}(z_s,z_c)=\int_0^{\pi}\frac{e^{ik_iD}}{D}\mathrm{d}\Phi \tag{42}$$

$$G_{4,i}(z_s,z_c)=\int_0^{\pi}(1-\cos\Phi)\frac{e^{ik_iD}}{D^2}\left(ik_i-\frac{1}{D}\right)\mathrm{d}\Phi \tag{43}$$

where D is defined by Eq. (20).

It should be noted that although the conditons of continuity of the pressure and the normal velocity on the duct extension were used in writing the integral expressions (37) and (38), this is not sufficient to take care of these conditions. The integral expressions (39) and (40) result from imposing these two conditions explicitly on the equations governing the sound field, whereas Eqs. (37) and (38) govern only surface quantities. For example, to show that Eqs. (37) and (38) are not sufficient to impose the condition of continuity of the acoustic radial velocity, we consider the case in which the free space Green's function $G(s,c)$ is replaced by another fundamental singularity of the Helmholtz equation, $H(s,c)$, such that

$$\frac{\partial H(s,c)}{\partial r_c} = 0 \tag{44}$$

In this case we have to use the general form of Stallybrass's result, Eq. (11). We notice that the additional term, which vanishes in the case of $G(s,c)$, does not include the normal derivative of the pressure. In effect, the new form of Eqs. (37) and (38) will not include any information about the normal derivative of the pressure. We conclude that the four equations (37-40) should be used for the determination of the unknown four surface functions.

We now proceed with this set of four integral expressions, according to the guidelines proposed in the solution of the Levine-Schwinger case, using a general collocation method. Guided with the argument in Sec. II regarding the success in using the Laguerre functions as trial functions, despite the existence of infinite velocities at the duct periphery ($r=1$, $z=0$) which exist also in the present problem, we assume a trial function expansion for each of the four unknown functions $p_i\ (z_s)$, $p_0\ (z_s)$, $p_-\ (z_s)$, and $\partial p_-\ (z_s)/\partial r_s$ as follows

$$\frac{\partial p_-(z)}{\partial r} = \sum_{n=1}^{N_1} B_n L_{n-1}(-k_1 z) \tag{45}$$

$$p_-(z) = \sum_{n=1}^{N_2} c_n L_{n-1}(-k_1 z) \tag{46}$$

$$p_i(z) = e^{-ik_1 z} + \sum_{n=1}^{N_3} D_n F_n(z) \tag{47}$$

$$p_0(z) = \sum_{n=1}^{N_4} E_n L_{n-1}(k_2 z) \tag{48}$$

with

$$F_1(z) = e^{ik_1 z}$$

$$F_n(z) = L_{n-2}(k_1 z) \qquad (n \geq 2)$$

where $L_n(z)$ is the Laguerre function of order n, and B_n, C_n, D_n, and E_n are the unknown expansion coefficients. The reflection coefficient is, by definition, equal to the coefficient D_1. The minus sign in the arguments of the Laguerre functions, Eqs. (45) and (46), exists because $p_-(z)$ and $\partial p_-(z)/\partial r$ are defined over the range from $z=-\infty$ to $z=0$, whereas the Laguerre functions are defined over the range from $z=0$ to $z=\infty$. These expansions are then substituted into Eqs. (37)-(40). By satisfying the expressions (37), (38), (39), and (40) at M_1, M_2, M_3, and M_4 collocation points, respectively, and by requiring the expansions to satisfy the conditions of pressure continuity at the open end of the duct $z=0$, namely

$$\sum_{n=1}^{N_2} C_n L_{n-1}(0) = 1 + \sum_{n=1}^{N_3} D_n F_n(0) \tag{49}$$

$$\sum_{n=1}^{N_2} C_n L_{n-1}(0) = \sum_{n=1}^{N_4} E_n L_{n-1}(0) \tag{50}$$

we obtain a total number of N equations, where

$$N = M_1 + M_2 + M_3 + M_4 + 2 \tag{51}$$

By choosing $N = N_1 + N_2 + N_3 + N_4$, we obtain a set of N algebraic equations in the N unknown coefficients. This set can be written in the following concise matrix notation

$$XA = Y \tag{52}$$

where A is the unknown coefficient column matrix.†

The same technique, described earlier is Sec. II, may be used to evaluate the singular double integrals occurring in the elements of X and Y. Finally, the solution of the set of Eqs. (52) leads to the knowledge of the reflection coefficient along with the rest of the expansion coefficients.

Results and Discussion

The same programming and numerical techniques, described in Sec. II were used to obtain the solution for the present case.

To recover the Levine-Schwinger case, as k_2 goes to k_1, the results were obtained with $N_3=8$ and $N_4=7$. In addition, Eqs. (37) and (38) were satisfied at the same seven collocation points. This allows the cancellation of all terms expressing $p_-(z)$ and $\partial p_-(z)/\partial r$ from this set of equations and from the conditions (49) and (50). This yields eight equations for the determination of the pressure discontinuity across the duct surface, which are the same as those given by Eq. (27).

The numerical solution was obtained for the case of $N_1=7$ and $N_2=7$. Thus, each unknown function was represented by a series expansion which contains Laguerre functions up to the sixth order. The numbers M_3 and M_4 were chosen according to some analytical properties of the four integral expressions. To this end, we consider the case in which the free space Green's function $G(s,c)$ is replaced by the fundamental singularity $H(s,c)$, defined by Eq. (44). In this case,

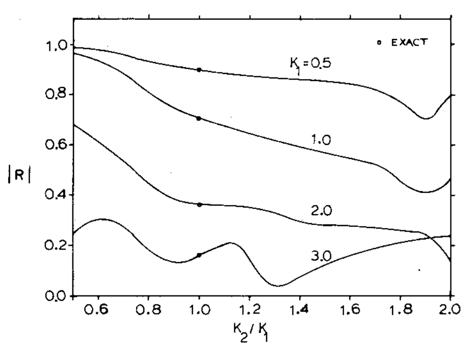

Fig. 3 $|R|$ **vs** k_2/k_1 **for several** k_1.

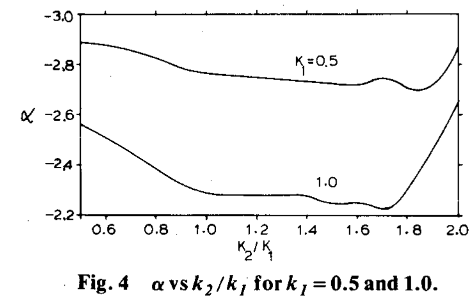

Fig. 4 α **vs** k_2/k_1 **for** $k_1 = 0.5$ **and 1.0.**

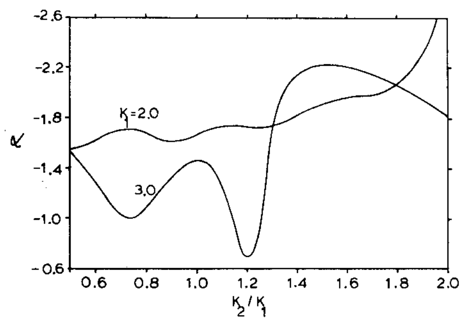

Fig. 5 α **vs** k_2/k_1 **for** $k_1 = 2.0, 3.0$.

†The details of the elements of this matrix equation are given in Ref. (17).

the new form of Eqs. (37), (38), and (39), along with the constraints (49) and (50), will not include the unknown function $\partial p_-(z)/\partial r$. Hence, to make the number of equations containing the other three unknown functions equal to the number of unknowns, we should have $M_4=6$. Then, for the same reason, $M_3=7$. In fact, it was found that satisfying the boundary condition of pressure continuity at the duct extension at seven points and satisfying the condition of normal velocity continuity on that surface at six points (i.e. $M_3=7$ and $M_4=6$) yields better convergence behavior for the expansion coefficients than the case with $M_3=6$ and $M_4=7$, at high frequencies and at large differences between k_1 and k_2.

The collocation points used in satisfying Eqs. (37) and (38) were equally distributed in the range $0<z_c\leq\lambda_{\text{inc}}$, where λ_{inc} is the wavelength of the incident wave. Similarly, the collocation points used in Eqs. (39) and (40) were equally distributed in the range $-\lambda_{\text{inc}}\leq z_c<0$.

It is worth mentioning here that when the collocation points in the range $-\lambda_{\text{inc}}\leq z_c<0$ are chosen to be the corresponding points to those in the range $0<z_c\leq\lambda_{\text{inc}}$, at the same distance from the origin, and when the argument of the Laguerre functions expressing the pressure on the outside surface of the duct is taken to be (k_1z) instead of (k_2z), then most of the integrals appearing in Eqs. (39) and (40), after substituting the series expansions, become equal to others in Eqs. (37) and (38). Also some of the integrals occurring in Eq. (40) become equivalent to others in Eq. (39). The computer time used to obtain the whole set of expansion coefficients for specific values of k_1 and k_2 using the above mentioned technique was 25 seconds on the CDC Cyber 74, whereas the time used when these integrals were computed independently from each other was 60 seconds.

The results are shown in Figs. 3 to 7 where the reflection coefficient is expressed as $R=|R|e^{i\alpha}$.

Practically, these figures show an increase in the magnitude of the reflection coefficient for the case of a cold core ($k_2<k_1$), and a decrease in its value for the case of hot core ($k_2>k_1$), as compared with the case of completely uniform medium ($k_2=k_1$) which is solved exactly by Levine and Schwinger.[3] Their values are indicated by small circles (o) on Fig. 3. Attempts to reason these results on a physical basis have not been successful. On the other hand, the figures do not indicate a similar general behavior for the phase change α, but a phase change of between $\pi/2$ and π is nearly always achieved, which indicates the tendency towards maintaining a constant pressure at the exit plane.

Although it is difficult to analytically obtain the limiting value of R as the frequency goes to zero, the figures shown an apparent convergence of $|R|$ toward the value 1 and of α toward the value π as k_1 goes to zero.

Figures 6 and 7 show the values of $|R|$ and α, as functions of the frequency of the incident wave, for a hot core case with $T_1/T_2=2$ ($k_2k_1=\sqrt{2}$), along with the corresponding values, obtained in the present analysis, for the completely uniform medium ($k_2=k_1$).

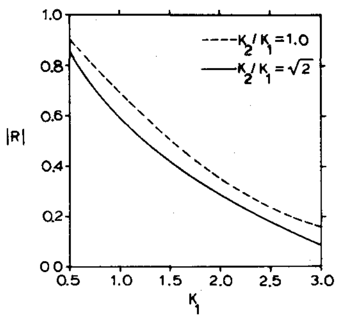

Fig. 6 $|R|$ **vs** k_1 **for** $k_2/k_1=$ **1.0 and** $\sqrt{2}$.

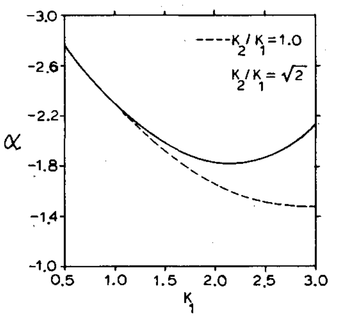

Fig. 7 α **vs** k_1 **for** $k_2/k_1=1.0$ **and** $\sqrt{2}$.

Conclusions

A new method for rapidly solving problems of sound radiation from a semi-infinite duct for the case of low Mach number has been developed and a new form of the Helmholtz integral formula is presented. The results obtained for the Levine-Schwinger problem agree well with the exact values. In the case of a duct flow having a temperature mismatch with the surroundings, the results show an increase in the magnitude of the reflection coefficient for the case of a cold core, and a decrease in its value for the case of a hot core, as compared with the case of exhaust into a uniform medium. A phase change of between $\pi/2$ and π is nearly always achieved, which indicates the tendency towards maintaining a constant pressure at the exit plane. The method can easily be extended to arbitrary duct shapes, as long as they are axisymmetric; and the method consumes relatively little computer time.

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References

[1]Strahle, W.C. and Shivashankara, B.N., "Combustion Generated Noise in Gas Turbine Combustors," NASA CR-134843, Aug. 1974.

[2]Noble, B., *Methods Based on the Wiener-Hopf Technique for the Solution of Partial Differential Equations,* Pergamon Press, New York, 1958.

[3]Levine, H. and Schwinger, J., "On the Radiation of Sound from an Unflanged Circular Pipe," *Physical Review,* Vol. 37, 1948, pp. 383-406.

[4]Carrier, G.F., "Sound Transmission from a Tube with Flow," *Quarterly of Applied Mathematics,* Vol. XIII, 1956, pp. 457-461.

[5]Lansing, D.L., Drischler, J.A., and Pusey, C. G., "Radiation of Sound from an Unflanged Circular Duct with Flow," presented at 79th Meeting of Acoustical Society of America, Atlantic City, New Jersey, 1970.

[6]Carrier, G.F., "Useful Approximations in Wiener-Hopf Problems, " *Journal of Applied Physics,* Vol. 30, 1959, pp. 1769-1774.

[7]Mani, R., "Refraction of Acoustic Duct Waveguide Modes by Exhaust Jets," *Quarterly of Applied Mathematics,* Vol. 30, 1973, pp. 501-520.

[8]Savkar, S.D. and Edelfelt, I.H., "Radiation of Cylindrical Duct Acoustic Modes with Flow Mismatch," proposed NASA-CR.

[9]Crandall, S.H., *Engineering Analysis,* McGraw-Hill, New York, 1956, pp. 150-153.

[10]Collatz, L., *The Numerical Treatment of Differential Equations,* Springer, Berlin, 1960, pp. 409-413.

[11]Fenlon, F.H., "Calculations of the Acoustic Radiation Field at the Surface of a Finite Cylinder by the Method of Weighted Residuals," *Proceedings of the IEEE,* Vol. 57, 1969, pp. 291-306.

[12]Butler, J.L., "Solution of Acoustical Radiation Problems by Boundary Collocation," *Journal of the Acoustical Society of America,* Vol. 48, 1970 pp. 325-336.

[13]Sommerfeld, A., *Partial Differential Equations in Physics,* Academic Press, New York, 1949, p. 189.

[14]Kellog, O.D., *Potential Theory,* Dover Publications, Inc., New York, 1961.

[15]Courant, R. and Hilbert, D., *Methods of Mathematical Physics,* Vol. 2, Interscience Publishers, New York, 1953, p. 256.

[16]Stallybrass, M.P., "On a Pointwise Variational Principle for the Approximate Solution of Linear Boundary Value Problems," *Journal of Mathematics and Mechanics*, Vol. 16, 1967, pp. 1247-1286.

[17]Wahbah, M.M., "Analysis Related to Combustion Noise Research," Ph.D Thesis, Georgia Institute of Technology, Atlanta, Georgia, 1975.

[18]Morse, P.M. and Ingard, K.U., *Theoretical Acoustics*, McGraw-Hill, New York, 1968, pp. 509-514.

[19]Courant, R. and Hilbert, D., *Methods of Mathematical Physics*, Vol. 1, Interscience Publishers, New York, 1953, pp. 93-97.

[20]Lamb, H., *Hydrodynamics*, Dover Publications, Inc., New York, 1945, pp. 106-108.